E9 203 Compressed Sensing & Sparse SP.

Last time:

- OMP, properties
- Thresholding based algos.
  - Basic thresholding.

Today:

- More thresholding based algos
- Reweighting-based algos

Recall notation: $z \in \mathbb{C}^N$

$L_s(z)$ = Index set of s largest absolute entries of z.

$H_s(z) = z_{L_s(z)}$ = Best s-term approx of z.

Basic Thresholding:

$$S^\# = L_s(A^H y)$$

$$x^\# = \underset{z \in \mathbb{C}^N}{\arg\min} \{ \|y - Az\|_2 ,\ \mathrm{supp}(z) \subseteq S^\# \}.$$

Iterative Hard Thresholding Algo:

Consider: $\underbrace{A^H A}_{N \times N} z = A^H y$

Fixed-point eqn: $z = (I - A^H A) z + A^H y$

Fixed-point iteration: $x^{(n+1)} = \underline{\underline{(I - A^H A) x^{(n)} + A^H y}}$

Init: $x^{(0)} = 0$

Iterate

$$x^{(n+1)} = H_s(x^{(n)} + A^H(y - Ax^{(n)}))$$

Until a stopping criterion is met at $\bar{n}$

output $x^\# = x^{(\bar{n})}$

Iterative hard thresholding.

Hard thresholding pursuit:

Init: $x^{(0)} = 0$

Iterate:

$$S^{(n+1)} = L_s(x^{(n)} + A^H(y - Ax^{(n)}))$$

$$x^{(n+1)} = \underset{z \in \mathbb{C}^N}{\arg\min} \{ \|y - Az\|_2 ,\ \mathrm{supp}(z) \subseteq S^{(n+1)} \}$$

Until a stopping criterion is met at $\bar{n}$

Output $x^\# = x^{(\bar{n})}$.

Compressive Sampling Matching Pursuit (CoSaMP):

Init: $x^{(0)} = 0$

Iterate:

$$\mathcal{U}^{(n+1)} = \mathrm{supp}(x^{(n)}) \cup \underline{L_{2s}(A^H(y - Ax^{(n)}))}$$

$$u^{(n+1)} = \underset{z \in \mathbb{C}^N}{\arg\min} \{ \|y - Az\|_2 ,\ \mathrm{supp}(z) \subseteq \mathcal{U}^{(n+1)} \}$$

$$x^{(n+1)} = H_{\underline{s}}(u^{(n+1)})$$

Until a stopping criterion is met at $\bar{n}$.

Output $x^\# = x^{(\bar{n})}$

Matching Pursuit:

$$x^{(n+1)} = x^{(n)} + t e_j ,\quad \underbrace{t \in \mathbb{C},\ j \in [N]}_{\text{Choose to min } \|y - Ax^{(n+1)}\|_2}$$

$$\Rightarrow \left.\begin{aligned} j &= \underset{\ell \in [N]}{\arg\max} \left| \underline{(A^H(y - Ax^{(n)}))_\ell} \right| \\ t &= \underline{(A^H(y - Ax^{(n)}))_j} \end{aligned}\right\} \text{HW}$$

Subspace pursuit:

Init: $x^{(0)} = 0$, $S^0 = [\phi]$.

Iterate

$$\mathcal{U}^{n+1} = S^n \cup L_{\underline{s}}(A^H(\underline{y - Ax^{(n)}}))$$

$$u^{n+1} = \underset{z \in \mathbb{C}^N}{\arg\min} \{ \|y - Az\|_2 ,\ \mathrm{supp}(z) \subseteq \mathcal{U}^{n+1} \}$$

$$\underset{\text{in CoSaMP.}}{H_s(u^{(n+1)})} \left\{\begin{aligned} S^{n+1} &= L_s(u^{n+1}) \\ x^{(n+1)} &= \underset{z \in \mathbb{C}^N}{\arg\min} \{ \|y - Az\|_2 ,\ \mathrm{supp}(z) \subseteq S^{n+1} \} \end{aligned}\right.$$

Until a stopping criterion is met at $\bar{n}$.

Output $x^\# = x^{(\bar{n})}$.

Soft thresholding pursuit: (When s is unknown):

Threshold τ, z_j: real valued

$$s_\tau(z_j) = \begin{cases} \mathrm{sgn}(z_j)(|z_j| - \tau) & \text{if } |z_j| \geq \tau \\ 0 & \text{else.} \end{cases}$$

Criteria for selecting algorithms:

1. Min. # meas. for recovery.
2. Speed
   - Low sparsity: OMP is fast
     - BT, MP are fast
     - CoSaMP, HTP are fast
   - ℓ_1-min depends on which algo is used to solve $(P_{1,\eta})$
     - Chambrolle & Pollock's primal-dual algo
       - Run-time roughly indep. of s
     - Homotopy based methods
       - Adds support one by one
       - Works better for small s.
3. Exploiting the structure of A / fast vector-matrix mult.
4. Numerical precision demanded by the algos.
5. Robustness to noise or when x is not exactly sparse.
6. Parameters that need to be hand-tuned.

Weighted ℓ_1-norm minimization

$W \in \mathbb{R}^{N \times N}$ diagonal, non-negative

$$\min_{z \in \mathbb{C}^N} \|W^{-1} z\|_1 \quad \text{s.t. } Az = y.$$

- Reweighting based algos
- Incorporating prior knowledge.

Has an optimal basic feasible soln. (cols. of A corresp. supp. of the soln, are LI)

Let $q \triangleq W^{-1} z$, $z = Wq$

$$\min_{q \in \mathbb{C}^N} \|q\|_1 \quad \text{s.t. } (A\underline{W}) q = y$$

$\Rightarrow x_0 = \underline{W} q_0$ is a BFS if q_0 is a BFS.

... measures (regularizers):

General diversity measure

$$\min_{z \in \mathbb{C}^N} \underline{\underline{G(z)}} \quad \text{s.t. } Az = y.$$

$$G(z) = \sum_{i=1}^{N} \underline{g_i}(|z_i|) \qquad \text{separable fn.}$$

often $g_i(\cdot) = g(\cdot)$ for simplicity

$g(|z|)$ is monotonically $\uparrow$, $g(0)$ is bounded.

$$g(x) = |x|^p, \quad p \geqslant 0$$

$$g(x) = \log |x|.$$

Recall: (Real-valued case): reformulate l_1 min as an LP:

$$z = z_+ - z_-, \quad z_+ \geqslant 0, \; z_- \geqslant 0, \quad w = \begin{bmatrix} z_+ \\ z_- \end{bmatrix} \in \mathbb{R}^{2N}$$

Result: (PG_1) and (PG_2) below are equivalent, in the sense that $\exists$ a 1-1 mapping betn their local minima. Further, the objective fns are equal at corresponding local minima.

$g(z) = |z|^p$
$g(|z|) = |z|^p$.

$$(PG_1): \min_{z \in \mathbb{R}^N} J(z) \triangleq \sum_{i=1}^{N} g(\underline{|z_i|}) \quad \text{s.t. } Az = y$$

$$(PG_2): \min_{w \in \mathbb{R}^{2N}} \underline{\underline{J_1(w)}} = \sum_{i=1}^{N} \left\{ \underline{g(|z_{+,i}|) + g(z_{-,i})} - g(0) \right\}$$

$$\text{s.t. } [A \;\; -A] \begin{bmatrix} z_+ \\ z_- \end{bmatrix} = y; \quad w = \begin{bmatrix} z_+ \\ z_- \end{bmatrix} \geqslant 0.$$

We will focus on (PG_2).

- $g(x)$ is strictly concave
- $[A \;\; -A] \begin{bmatrix} z_+ \\ z_- \end{bmatrix} = y$: convex set K  polygon.

  Extreme pts: Ones that cannot be written as a convex linear combination of distinct pts. $\in K$.